UNIT 17: Changing the subject of formulae (more complex), algebraic fractions, solving equations arising from algebraic fractions, rationalising surds, proof	**Teaching time** 7–9 hours

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SPECIFICATION REFERENCES

N8 … **simplify surd expressions involving squares (e.g. √12 = √(4 × 3) = √4 × √3 = 2√3) and rationalise denominators**

A4 simplify and manipulate algebraic expressions (including those involving surds **and algebraic fractions**) by:

- collecting like terms
- multiplying a single term over a bracket
- taking out common factors
- expanding products of two **or more** binomials
- factorising quadratic expressions of the form $x^2 + bx + c$, including the difference of two squares; **factorising quadratic expressions of the form $ax^2 + bx + c$**
- simplifying expressions involving sums, products and powers, including the laws of indices

A5 … rearrange formulae to change the subject

A6 … argue mathematically to show algebraic expressions are equivalent, and use algebra to support and construct arguments **and proofs**

A7 where appropriate, interpret simple expressions as functions with inputs and outputs; **interpret the reverse process as the 'inverse function'; interpret the succession of two functions as a 'composite function' (the use of formal function notation is expected)**

A18 solve quadratic equations **(including those that require rearrangement)** algebraically by factorising, …

PRIOR KNOWLEDGE

Students should be able to simplify surds.

Students should be able to use negative numbers with all four operations.

Students should be able to recall and use the hierarchy of operations.

KEYWORDS

Tier 2

Rational, irrational, rearrange, subject, proof, notation, evaluate

Tier 3

Rationalise, denominator, surd, fraction, equation, inverse

OBJECTIVES

By the end of the unit, students should be able to:

- Rationalise the denominator involving surds;
- Simplify algebraic fractions;
- Multiply and divide algebraic fractions;
- Solve quadratic equations arising from algebraic fraction equations;
- Change the subject of a formula, including cases where the subject occurs on both sides of the formula, or where a power of the subject appears;
- Change the subject of a formula such as , where all variables are in the denominators;
- Solve 'Show that' and proof questions using consecutive integers (n, $n + 1$), squares a^2, b^2, even numbers $2n$, odd numbers $2n + 1$;
- Use function notation;
- Find f(x) + g(x) and f(x) – g(x), 2f(x), f(3x) etc algebraically;
- Find the inverse of a linear function;
- Know that $f^{-1}(x)$ refers to the inverse function;
- For two functions f(x) and g(x), find gf(x).

POSSIBLE SUCCESS CRITERIA/EXAM QUESTIONS

Rationalise: , , (√18 + 10) +√2.

Explain the difference between rational and irrational numbers.

Given a function, evaluate f(2).

When g(x) = 3 – 2x, find $g^{-1}(x)$.

Show that $\dfrac{1}{1+\dfrac{1}{\sqrt{2}}}$ can be written as $2 - \sqrt{2}$

(Total 3 marks)

New SAMs Paper 1H qu.23 (N8 – AO2)

Martin expands $(2x + 1)(2x - 3)(3x + 2)$

He gets $12x^3 - 4x^2 - 17x + 6$

Explain why Martin's solution cannot be correct.

(Total 1 mark)

Mock Papers Set 1, Paper 1H qu.16b (A4 – AO3)

Show that $\frac{1}{6x^2+7x-5} \div \frac{1}{4x^2-1}$ simplifies to $\frac{ax+b}{cx+d}$ where a, b, c and d are integers.

(Total 3 marks)

New SAMs Paper 2H qu.16 (A4 – AO2)

Make a the subject of $a+3=\frac{2a+7}{r}$

(Total 3 marks)

New SAMs Paper 1H qu.17 (A5 – AO1)

The product of two consecutive positive integers is added to the larger of the two integers.

Prove that the result is always a square number.

(Total 3 marks)

Specimen Papers Set 2, Paper 3H qu.17 (A6 – AO2)

$f(x) = x^3$
$g(x) = 4x - 1$

(*a*) Find fg(2)

(2)

$h(x) = fg(x)$

(*b*) Find an expression for $h^{-1}(x)$

$h^{-1}(x) =$ ....

(3)

(Total 5 marks)

Mock Papers Set 3, Paper 3H qu.21 (A7 – AO1)

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f and g are functions such that

$$f(x) = 3x^2 \text{ and } g(x) = \frac{1}{x-2}$$

Find gf(4).
Give your answer as a fraction.

(Total 2 marks)

Mock Papers Set 2, Paper 2H qu.21 (A7 – AO1)

Steve is asked to solve the equation $5(x + 2) = 47$
Here is his working.

$$\begin{aligned} 5(x + 2) &= 47 \\ 5x + 2 &= 47 \\ 5x &= 45 \\ x &= 9 \end{aligned}$$

Steve's answer is wrong.

(*a*) What mistake did he make?

(1)

Liz is asked to solve the equation $3x^2 + 8 = 83$

Here is her working.

$$\begin{aligned} 3x^2 + 8 &= 83 \\ 3x^2 &= 75 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

(*b*) Explain what is wrong with Liz's answer.

(1)

(Total 2 marks)

Specimen Papers Set 2, Paper 2H qu.8 (A18, A17 –AO2/AO3)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Formal proof is an ideal opportunity for students to provide a clear logical chain of reasoning providing links with other areas of mathematics.

COMMON MISCONCEPTIONS

√3 x √3 = 9 is often seen.

When simplifying involving factors, students often use the 'first' factor that they find and not the LCM.

NOTES

It is useful to generalise $\sqrt{m} \times \sqrt{m} = m$.

Revise the difference of two squares to show why we use, for example, (√3 – 2) as the multiplier to rationalise (√3 + 2).

Link collecting like terms to simplifying surds (Core 1 textbooks are a good source for additional work in relation to simplifying surds).

Practice factorisation where the factor may involve more than one variable.

Emphasise that, by using the LCM for the denominator, the algebraic manipulation is easier.

UNIT 18: Vectors and geometric proof	Teaching time 9–11 hours

SPECIFICATION REFERENCES

G25 apply addition and subtraction of vectors, multiplication of vectors by a scalar, and diagrammatic and column representations of vectors; **use vectors to construct geometric arguments and proof**

PRIOR KNOWLEDGE

Students will have used vectors to describe translations and will have knowledge of Pythagoras' Theorem and the properties of triangles and quadrilaterals.

KEYWORDS

Tier 2 - Proof

Tier 3 -Vector, direction, magnitude, scalar, multiple, parallel, collinear, ratio, column vector

OBJECTIVES

By the end of the unit, students should be able to:

- Understand and use vector notation, including column notation, and understand and interpret vectors as displacement in the plane with an associated direction.
- Understand that 2**a** is parallel to **a** and twice its length, and that **a** is parallel to –**a** in the opposite direction.
- Represent vectors, combinations of vectors and scalar multiples in the plane pictorially.
- Calculate the sum of two vectors, the difference of two vectors and a scalar multiple of a vector using column vectors (including algebraic terms).
- Find the length of a vector using Pythagoras' Theorem.
- Calculate the resultant of two vectors.
- Solve geometric problems in 2D where vectors are divided in a given ratio.
- Produce geometrical proofs to prove points are collinear and vectors/lines are parallel.

POSSIBLE SUCCESS CRITERIA/EXAM QUESTIONS

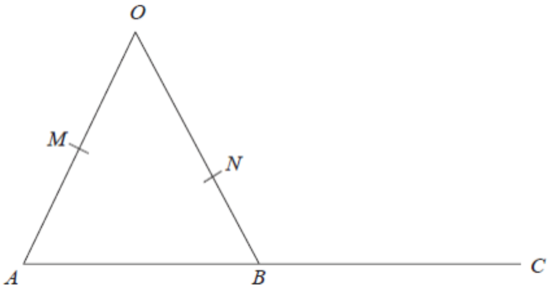
Add and subtract vectors algebraically and use column vectors.

Solve geometric problems and produce proofs.

$$\mathbf{a} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Work out **b** – 2**a** as a column vector.

(Total 2 marks)
New SAMs Paper 1F qu.30 (G25 – AO1)



OMA, *ONB* and *ABC* are straight lines.
M is the midpoint of *OA*.
B is the midpoint of *AC*.

$\overrightarrow{OA} = 6\mathbf{a}$ $\quad \overrightarrow{OB} = 6\mathbf{b}$ $\quad \overrightarrow{ON} = k\mathbf{b}$ where k is a scalar quantity.

Given that *MNC* is a straight line, find the value of k.

(Total 5 marks)
New SAMs Paper 3H qu.18 (G25, A4 – AO1/AO3)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

"Show that"-type questions are an ideal opportunity for students to provide a clear logical chain of reasoning providing links with other areas of mathematics, in particular algebra.

Find the area of a parallelogram defined by given vectors.

COMMON MISCONCEPTIONS

Students find it difficult to understand that parallel vectors are equal as they are in different locations in the plane.

NOTES

Students find manipulation of column vectors relatively easy compared to pictorial and algebraic manipulation methods – encourage them to draw any vectors they calculate on the picture. Geometry of a hexagon provides a good source of parallel, reverse and multiples of vectors.

Remind students to underline vectors or use an arrow above them, or they will be regarded as just lengths.

Extend geometric proofs by showing that the medians of a triangle intersect at a single point.

3D vectors or **i**, **j** and **k** notation can be introduced and further extension work can be found in GCE Mechanics 1 textbooks.

UNIT 19: Direct and indirect proportion: using statements of proportionality, reciprocal and exponential graphs, rates of change in graphs, functions, transformations of graphs

SPECIFICATION REFERENCES

A7 where appropriate, interpret simple expressions as functions with inputs and outputs; …

A12 recognise, sketch and interpret graphs of the reciprocal function with $x \neq 0$, **exponential functions $y = k^x$ for positive values of k** …

A13 **sketch translations and reflections of a given function**

A14 plot and interpret reciprocal graphs and **exponential graphs** …

A15 **calculate or estimate gradients of graphs and areas under graphs (including quadratic and other non-linear graphs) and interpret results in cases such distance–time graphs, velocity–time graphs and graphs in financial contexts (this does not include calculus)**

A21 translate simple situations or procedures into algebraic expressions or formulae; …

R10 solve problems involving direct and inverse proportion, including graphical and algebraic representations

R11 use compound units such as speed, rates of pay, unit pricing, density and pressure

R13 understand that X is inversely proportional to Y is equivalent to X is proportional to ; **construct and** interpret equations that describe direct and inverse proportion

R14 interpret the gradient of a straight line graph as a rate of change; recognise and interpret graphs that illustrate direct and inverse proportion

R15 **interpret the gradient at a point on a curve as the instantaneous rate of change; apply the concepts of average and instantaneous rate of change (gradients of chords and tangents) in numerical, algebraic and graphical contexts (this does not include calculus**

R16 set up, solve and interpret the answers in growth and decay problems …

PRIOR KNOWLEDGE

Students should be able to draw linear and quadratic graphs.

Students should be able to calculate the gradient of a linear function between two points.

Students should recall transformations of trigonometric functions.

Students should have knowledge of writing statements of direct proportion and forming an equation to find values.

KEYWORDS

Tier 2 Direct, indirect, estimate, area, distance, time, transformation

Tier 3 Reciprocal, linear, gradient, quadratic, exponential, proportion, rate of change, velocity, cubic, constant of proportionality

19a. Reciprocal and exponential graphs; Gradient and area under graphs (R11, R14, R15, R16, A7, A12, A13, A14, A15)	Teaching time 7–9 hours

OBJECTIVES

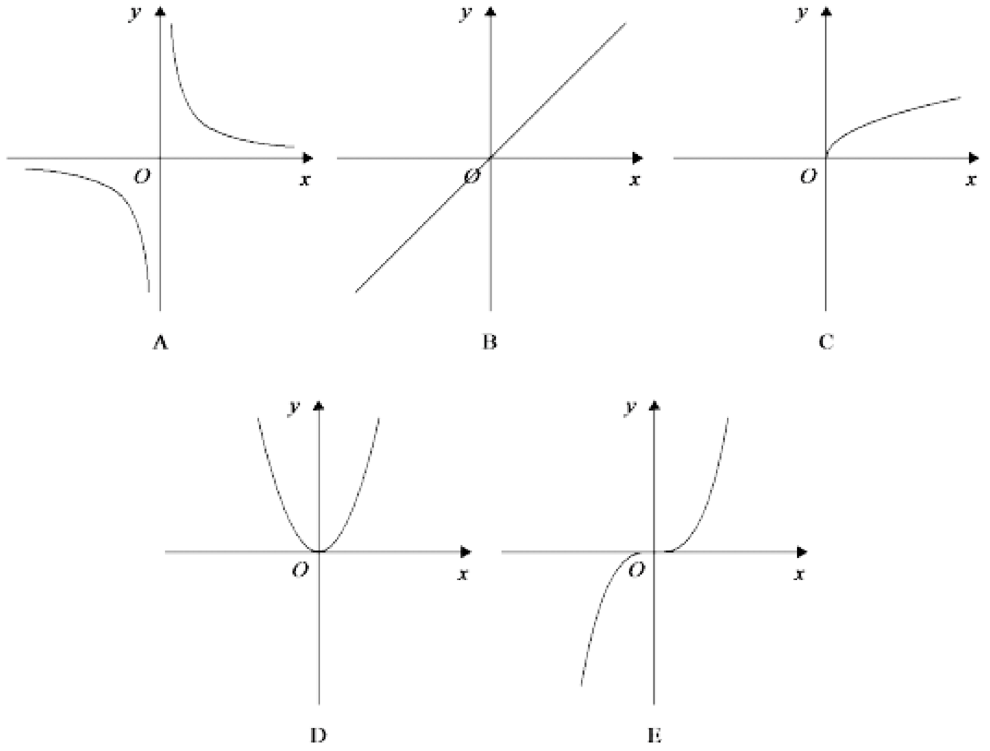
By the end of the sub-unit, students should be able to:

- Recognise, sketch and interpret graphs of the reciprocal function with $x \neq 0$
- State the value of x for which the equation is not defined;
- Recognise, sketch and interpret graphs of exponential functions $y = k^x$ for positive values of k and integer values of x;
- Use calculators to explore exponential growth and decay;
- Set up, solve and interpret the answers in growth and decay problems;
- Interpret and analyse transformations of graphs of functions and write the functions algebraically, e.g. write the equation of $f(x) + a$, or $f(x - a)$:
  - apply to the graph of $y = f(x)$ the transformations $y = -f(x)$, $y = f(-x)$ for linear, quadratic, cubic functions;
  - apply to the graph of y = $f(x)$ the transformations $y = f(x) + a$, $y = f(x + a)$ for linear, quadratic, cubic functions;
- Estimate area under a quadratic or other graph by dividing it into trapezia;
- Interpret the gradient of linear or non-linear graphs, and estimate the gradient of a quadratic or non-linear graph at a given point by sketching the tangent and finding its gradient;
- Interpret the gradient of non-linear graph in curved distance–time and velocity–time graphs:
  - for a non-linear distance–time graph, estimate the speed at one point in time, from the tangent, and the average speed over several seconds by finding the gradient of the chord;
  - for a non-linear velocity–time graph, estimate the acceleration at one point in time, from the tangent, and the average acceleration over several seconds by finding the gradient of the chord;
- Interpret the gradient of a linear or non-linear graph in financial contexts;
- Interpret the area under a linear or non-linear graph in real-life contexts;
- Interpret the rate of change of graphs of containers filling and emptying;
- Interpret the rate of change of unit price in price graphs.

POSSIBLE SUCCESS CRITERIA/EXAM QUESTIONS

Explain why you cannot find the area under a reciprocal or tan graph.

Here are five graphs.
Each graph shows either direct proportion or inverse proportion.



The table shows five equations.

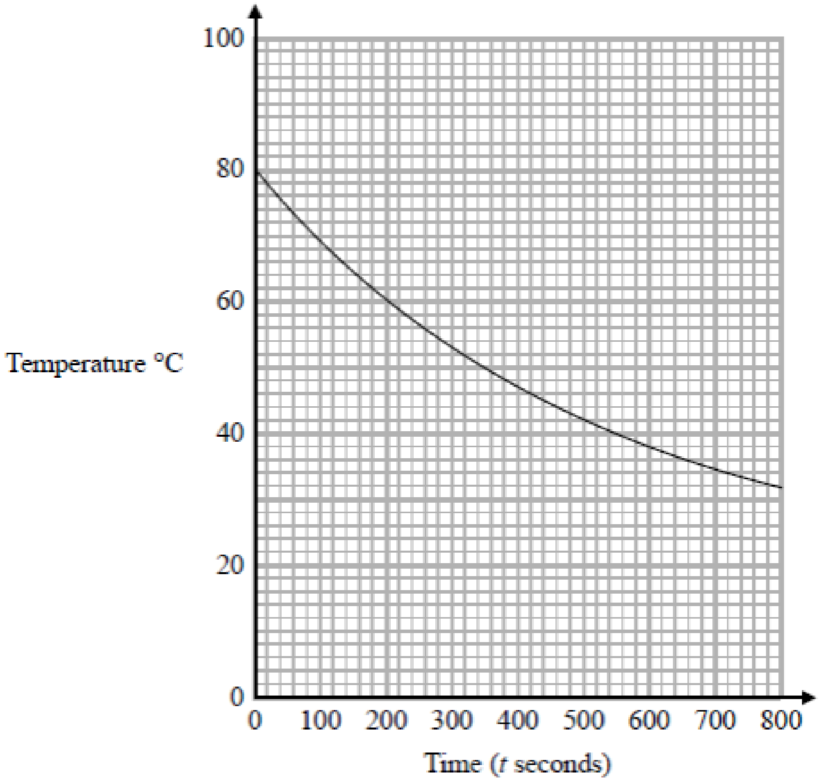
Equation	Graph
$y = kx^3$	....
$y = k\sqrt{x}$	....
$y = kx^2$	....
$y = \frac{k}{x}$	....
$y = kx$	....

Match the letter of each graph to its equation.

(Total 3 marks)

Mock Papers Set 3, Paper 2H qu.13 (R14 – AO2)

The graph gives information about the variation in the temperature, in °C, of an amount of water that is allowed to cool from 80 °C.



(*b*) (i) Work out the average rate of decrease of the temperature of the water between $t = 0$ and $t = 800$.

The instantaneous rate of decrease of the temperature of the water at time T seconds is equal to the average rate of decrease of the temperature of the water between $t = 0$ and $t = 800$.

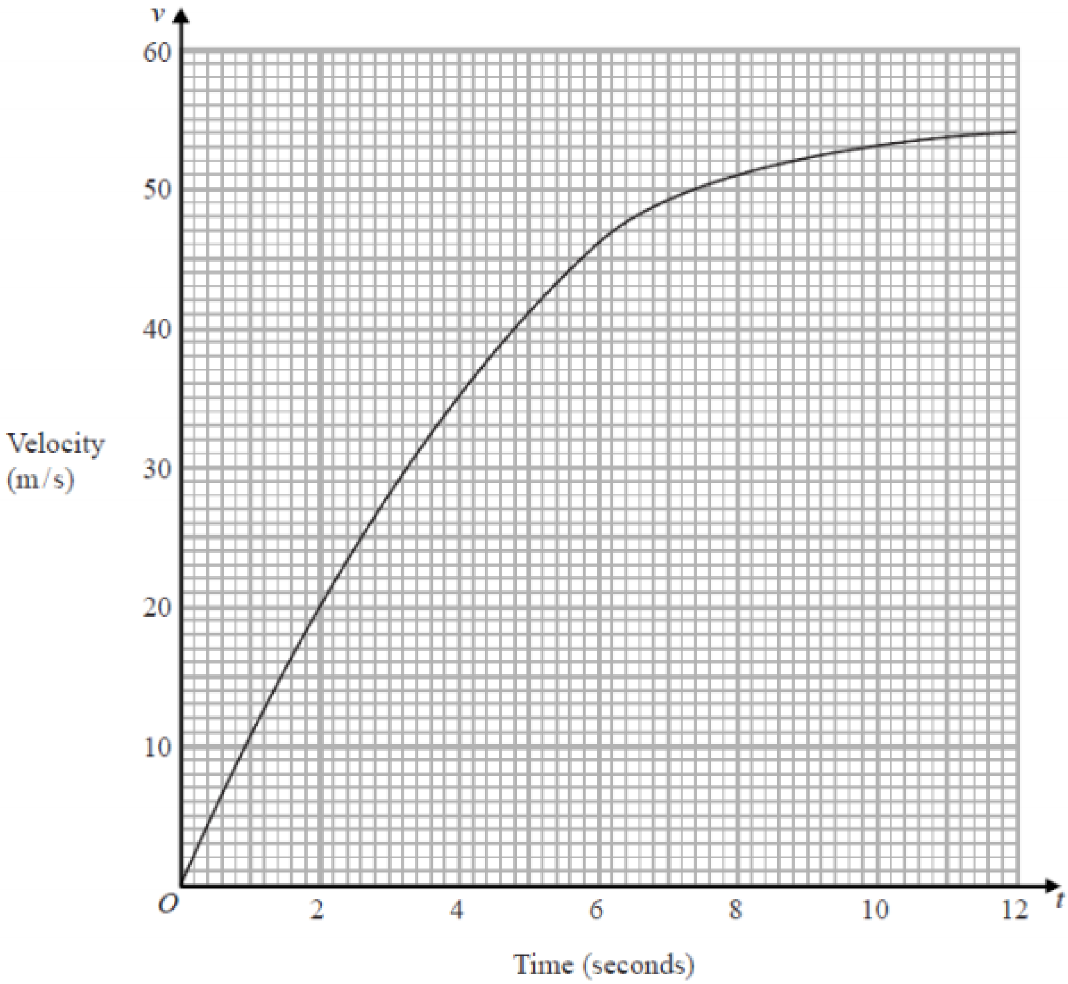
(ii) Find an estimate for the value of T.
You must show how you got your answer.

(4)

(Total 6 marks)

Original SAMs Paper 3H qu.14 (R15, R14, A5 – AO1/AO2/AO3)

The graph shows information about the velocity, v m/s, of a parachutist t seconds after leaving a plane.



(*a*) Work out an estimate for the acceleration of the parachutist at $t = 6$

(2)

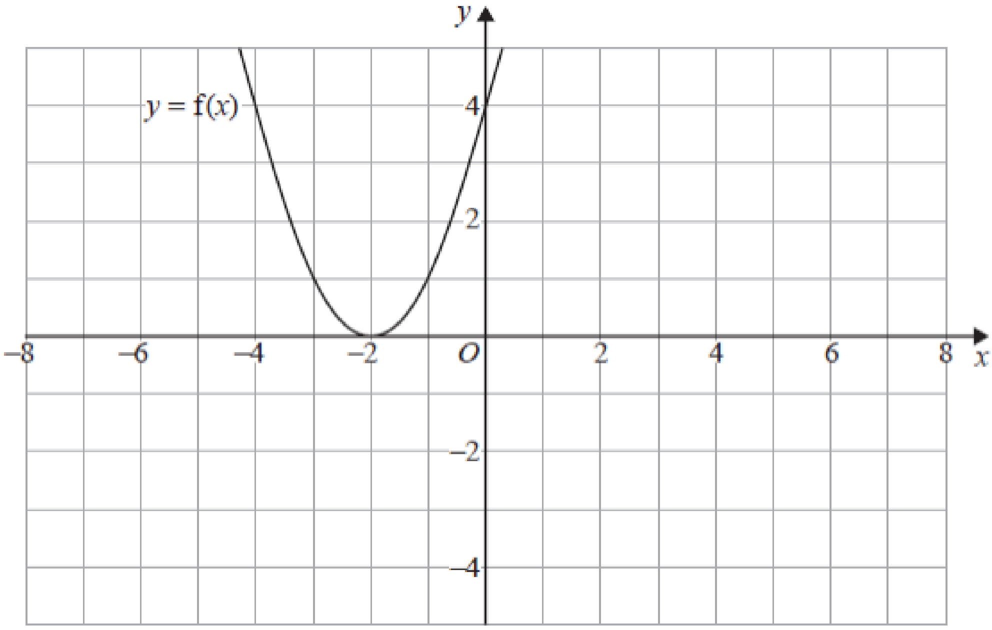
(*b*) Work out an estimate for the distance fallen by the parachutist in the first 12 seconds after leaving the plane.
Use 3 strips of equal width.

(3)

(Total 5 marks)

Specimen Papers Set 1, Paper 2H qu.20 (A15 –AO1/AO2)

The graph of $y = f(x)$ is shown on both grids below.



(*a*) On the grid above, sketch the graph of $y = f(-x)$

(1)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Interpreting many of these graphs in relation to their specific contexts.

COMMON MISCONCEPTIONS

The effects of transforming functions is often confused.

NOTES

Formal function notation along with inverse and composite functions will have been encountered but are topics that students may need to be reminded about.

Translations and reflections of functions are included in this specification, but not rotations or stretches.

Financial contexts could include percentage or growth rate.

When interpreting rates of change with graphs of containers filling and emptying, a steeper gradient means a faster rate of change.

When interpreting rates of change of unit price in price graphs, a steeper graph means larger unit price.

19b. Direct and inverse proportion (A21, R7, R10, R11, R14)	**Teaching time** 7–9 hours

OBJECTIVES

By the end of the sub-unit, students should be able to:

- Recognise and interpret graphs showing direct and inverse proportion;
- Identify direct proportion from a table of values, by comparing ratios of values, for x squared and x cubed relationships;
- Write statements of proportionality for quantities proportional to the square, cube or other power of another quantity;
- Set up and use equations to solve word and other problems involving direct proportion;
- Use $y = kx$ to solve direct proportion problems, including questions where students find k, and then use k to find another value;
- Solve problems involving inverse proportion using graphs by plotting and reading values from graphs;
- Solve problems involving inverse proportionality;
- Set up and use equations to solve word and other problems involving direct proportion or inverse proportion.

POSSIBLE SUCCESS CRITERIA/EXAM QUESTIONS

Understand that when two quantities are in direct proportion, the ratio between them remains constant.

Know the symbol for 'is proportional to'.

d is inversely proportional to c

When $c = 280$, $d = 25$

Find the value of d when $c = 350$

(Total 3 marks)

New SAMs Paper 2H qu.13 (R13, R10 – AO1)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Justify and infer relationships in real-life scenarios to direct and inverse proportion such as ice cream sales and sunshine. **COMMON MISCONCEPTIONS**

Direct and inverse proportion can get mixed up.

NOTES

Consider using science contexts for problems involving inverse proportionality, e.g. volume of gas inversely proportional to the pressure or frequency is inversely proportional to wavelength.